Nov 22

Lecture

(I) surfaces (cont'd)
(II) surface integrals of functions
(III) surface integral of vector fields.

(I) Examples of surfaces

Our last class example of surfaces are surfaces of revolution.

Let $x(t)\hat{i} + z(t)\hat{k}$ be a curve, $t \in [a,b]$, in the right half xz-plane. Rotate it around the z-axis to get a surface S. The standard parametrization of S:

$$\underset{[a,b]\times[0,2\pi]}{(t,\alpha)} \longmapsto x(t)\cos\alpha\,\hat{i} + x(t)\sin\alpha\,\hat{j} + z(t)\hat{k}$$

$$\vec{r}_t = x'\cos\alpha\,\hat{i} + x'\sin\alpha\,\hat{j} + z'\hat{k}$$

$$\vec{r}_\alpha = -x\sin\alpha\,\hat{i} + x\cos\alpha\,\hat{j} + 0\hat{k}$$

$$\vec{r}_t \times \vec{r}_\alpha = -xz'\hat{i} - xz'\hat{j} + x'x\hat{k}$$

$$|\vec{r}_t \times \vec{r}_\alpha| = x(x'^2 + z'^2)^{\frac{1}{2}},$$

so if the curve is regular, then S is also regular.

(II) Surface integrals of Functions

Let $\vec{r}(u,v)$, $(u,v) \in D$, be a parametrization of a surface S. Let $f = f(x,y,z)$ be a continuous function on S. We will define the surface integral of f over S to be

$$\iint_S f(x,y,z)\,d\sigma = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA(u,v)$$

so that

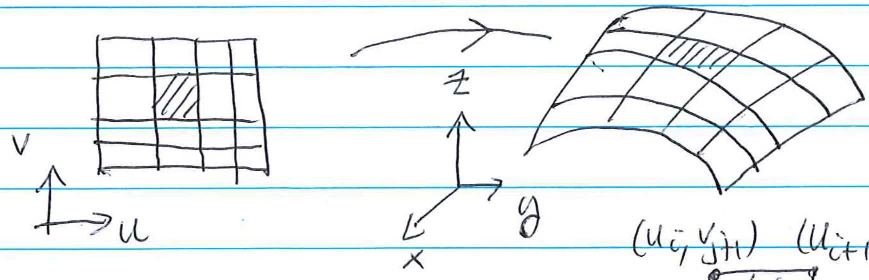
- when $f \equiv 1$, $\iint_S d\sigma = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA(u,v)$ gives

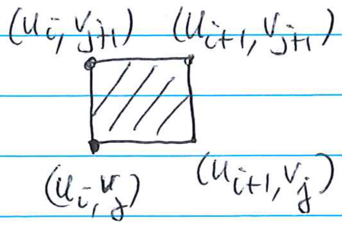
the surface area of S,

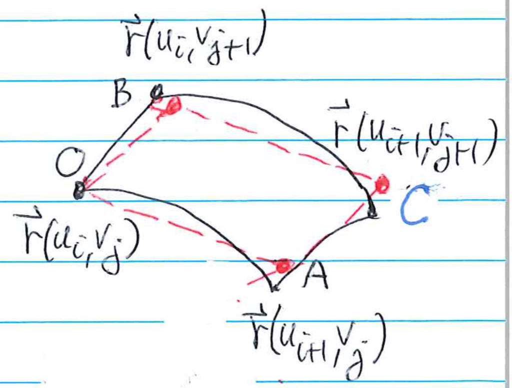
- when $f \geq 0$, $\iint_S f\,d\sigma$ give the mass of the thin

object S with density function f.

Motivation. Let D be a rectangle for simplicity. A partition P in D introduces a corresponding "partition" on S, that is, breaking it up into many small pieces of surfaces.







By Taylor's expansion,

A $(\vec{r}(u_i,v_j) + \vec{r}_u(u_i,v_j)\Delta u)$

B $(\vec{r}(u_i,v_j) + \vec{r}_v(u_i,v_j)\Delta v)$

$$\vec{r}(u_{i+1}, v_j) = \vec{r}(u_i, v_j) + \vec{r}_u(u_i, v_j)\Delta u + \text{higher order terms}$$

$$\vec{r}(u_i, v_{j+1}) = \vec{r}(u_i, v_j) + \vec{r}_v(u_i, v_j)\Delta v + \text{higher order terms}$$

Ignoring the higher order terms, the small piece is approximated by the parallelogram at OABC. Then the area of the small piece is approximately equal to the area of the parallelogram.

whose area is $\left|\vec{r}_u \times \vec{r}_v\right|(u_i, v_j)\,\Delta u_i \Delta v_j$.

$\therefore$ The "Riemann sum" of the surface integral is

$$\sum_{i,j} f(\text{tag pts})\left|\vec{r}_u \times \vec{r}_v\right| \Delta u_i \Delta v_j$$

Let $\|P\| \to 0$, get

$$\iint_D f(\vec{r}(u,v))\left|\vec{r}_u \times \vec{r}_v\right|(u,v)\, dA(u,v).$$

e.g. 1. Find the surface area of the sphere

$$x^2+y^2=a^2.$$

A standard parametrization of the sphere is

$$(\varphi, \theta) \longmapsto a\sin\varphi\cos\theta\,\hat{i} + a\sin\varphi\sin\theta\,\hat{j} + a\cos\varphi\,\hat{k}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = a^2\sin^2\varphi\cos\theta\,\hat{i} + a^2\sin^2\varphi\cos\theta\,\hat{j} + a^2\sin\varphi\cos\varphi\,\hat{k}$$

$$\left|\vec{r}_\varphi \times \vec{r}_\theta\right| = a^2\sin\varphi.$$

$$\therefore \text{surface area} = \iint_{[0,\pi]\times[0,2\pi]} \left|\vec{r}_\varphi \times \vec{r}_\theta\right| dA(\varphi,\theta)$$

$$= a^2\int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\theta$$

$$= 4\pi a^2. \quad \#$$

e.g. 2 The surface S is cut from $z = x^2+y^2$, at $z=0,4$. Find its surface area.

$z=4$ and $z=x^2+y^2$ cut at $x^2+y^2=4$, ie, D_2.

standard parametrizatia for a graph:

$$\underset{D_2}{(x,y)} \longmapsto (x, y, x^2+y^2)$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+f_x^2+f_y^2} = \sqrt{1+4(x^2+y^2)}$$

$$\therefore \text{surface area} = \iint_{D_2} \sqrt{1+4(x^2+y^2)}\, dA(x,y)$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\, r\,dr\,d\theta \quad (\text{polar coor.})$$

$$\vdots$$

$$= \frac{\pi}{6}\left(17^{3/2}-1\right) \quad \#$$

e.g. 3 Rotate the curve $x=\cos z$, $z\in[-\pi/2, \pi/2]$, around the z-axis to get S. Find its surface area.

The general formula for S is

$$(t,\alpha) \longmapsto (x(t)\cos\alpha,\ x(t)\sin\alpha,\ z(t))$$

Here, it becomes

$$\underset{[-\pi/2,\pi/2]\times[0,2\pi]}{(z,\alpha)} \longmapsto (\cos z\cos\alpha,\ \cos z\sin\alpha,\ z)$$

$$\vec{r}_z = -\sin z\cos\alpha\,\hat{i} - \sin z\sin\alpha\,\hat{j} + \hat{k}$$

$$\vec{r}_\alpha = -\cos z\sin\alpha\,\hat{i} + \cos z\cos\alpha\,\hat{j} + 0\,\hat{k}.$$

$$\vec{r}_z \times \vec{r}_\alpha = -\cos\alpha\cos z\,\hat{i} - \sin\alpha\cos z\,\hat{j} - \cos z\sin z\,\hat{k}$$

$$|\vec{r}_z \times \vec{r}_\alpha| = \cos z\sqrt{1+\sin^2 z}$$

$$\therefore \text{Surface area} = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos z\sqrt{1+\sin^2 z}\,dz\,d\alpha$$

$$= 2\pi\int_{-\pi/2}^{\pi/2} \cos z\sqrt{1+\sin^2 z}\,dz$$

$$= 4\pi\int_0^{\pi/2} \cos z\sqrt{1+\sin^2 z}\,dz$$

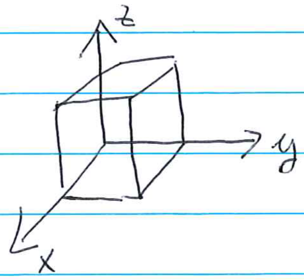
$$= 4\pi\int_0^1 \sqrt{1+t^2}\,dt$$

$$\vdots$$

$$= 2\pi\left[\sqrt{2} + \ln(1+\sqrt{2})\right]. \quad \#$$

e.g. 4 Let C be the unit cube in the octant with one vertex at the origin. Find

$$\iint_C xyz\,d\sigma.$$



C is composed of 6 faces.
The faces at $x=0$, $y=0$, $z=0$, have no contribution to the integral as $F(x,y,z) = xyz = 0$ there. So

$$\iint_C xyz\,d\sigma = \iint_{\substack{\text{face}\\ \text{at } x=1}} xyz\,d\sigma + \iint_{\substack{\text{face}\\ \text{at } y=1}} xyz\,d\sigma + \iint_{\substack{\text{face}\\ \text{at } z=1}} xyz\,d\sigma$$

$$= 3\iint_{\substack{\text{face}\\ \text{at } z=1}} xyz\,d\sigma \quad (\text{by symmetry})$$

The face at $z=1$ has a parametrization

$$(x,y) \longmapsto (x,y,1)$$
$$[0,1]\times[0,1]$$

It is a graph $f(x,y)\equiv 1$, so

$$|\vec{r}_x\times\vec{r}_y| = \sqrt{1+f_x^2+f_y^2} = 1$$

$$\therefore \iint_{\substack{\text{face}\\ \text{at } z=1}} xy\,1\times 1\, dA(x,y) = \int_0^1\int_0^1 xy\, dy\, dx = \frac{1}{4}$$

$$\therefore \iint_C xyz\, d\sigma = 3/4 \quad \#$$

(VI) Surface Integrals of Vector Fields.

Let $\vec{r}(u,v)$ be a parametrization of a surface S. The vector

$$\vec{r}_u\times\vec{r}_v$$

satisfies $\vec{r}_u\times\vec{r}_v\cdot\vec{r}_u=0$, $\vec{r}_u\times\vec{r}_v\cdot\vec{r}_v=0$, hence points in the normal direction. We define the unit vector

$$\hat{n} = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$$

to be the unit normal vector field of the parametrization $\vec{r}$. It is clear that all parametrizations of S can be divided into 2 classes, whose unit normal vector fields pointing in opposite direction.

A surface with a chosen unit normal vector field is called an <u>oriented surface</u>.

Just like we integrate a vector field along an oriented curve, we are going to define the surface integral of a v.f. over an oriented surface.

Let $\vec{F}$ be a v.f. defined on the surface S and let $\vec{r}: D \to S$ be an admissible parametrization so that the normal vector $\hat{n}$ is given by

$$\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

We define the surface integral of $\vec{F}$ over S to be

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma.$$

Using

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA(u,v)$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA(u,v),$$

which is the formula to evaluating the integral.

$\vec{F} \cdot \hat{n}$ is the projection of $\vec{F}$ onto the direction $\hat{n}$.

Hence

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

give the <u>flux</u> of $\vec{F}$ across S in the direction of $\hat{n}$.

e.g 5. Let S be the graph of $y = x^2$ over $[0,1]\times[0,4]$ in the xz-plane where normal is chosen to pointing to the $-y$-direction. Evaluate

$$\iint_S (yz\hat{i} + x\hat{j} - z^2\hat{k}) \cdot \hat{n}\, d\sigma$$

Using x, z as parameters

$$(x,z) \longmapsto (x, x^2, z)$$
$$[0,1]\times[0,4]$$

$$\vec{r}_x = (1, 2x, 0) = \hat{i} + 2x\hat{j} + 0\hat{k}$$
$$\vec{r}_z = (0, 0, 1) = 0\hat{i} + 0\hat{j} + \hat{k}$$

$\vec{r}_x \times \vec{r}_z = 2x\hat{i} - \hat{j} + 0\hat{k} = (2x, -1, 0)$ the y-component $-1 < 0$

$\therefore$ $\vec{r}_x \times \vec{r}_z$ points in the $-y$-direction and

$$\hat{n} = \frac{2x\hat{i} - \hat{j} + 0\hat{k}}{|\vec{r}_x \times \vec{r}_z|}$$

$$\iint_S (yz\hat{i} + x\hat{j} - z^2\hat{k}) \cdot \hat{n}\, d\sigma = \iint_D (x^2z\hat{i} + x\hat{j} - z^2\hat{k}) \cdot \vec{r}_x \times \vec{r}_z \, dA(x,z)$$

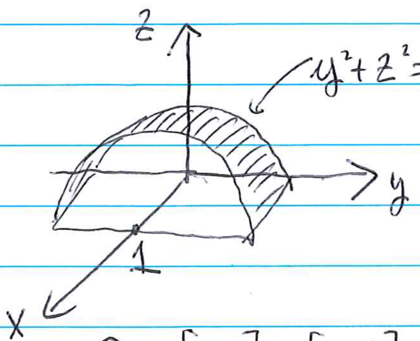
$$= \iint_D (x^2z\hat{i} + x\hat{j} - z^2\hat{k}) \cdot (2x, -1, 0)\, dA(x,z)$$

$$= \iint_D (2x^3z - x)\, dA(x,z)$$

$$= \int_0^1 \int_0^4 (2x^3z - x)\, dz\, dx$$

$$= 2 \quad \#$$

e.g. 6 Find the flux of $\vec{F} = yz\hat{j} + z^3\hat{k}$ outward through the surface S cut from the cylinder $y^2+z^2=1$, $z\geq 0$, by the plane $x=0$, $x=1$.



$y^2+z^2=1 \Rightarrow z=\sqrt{1-y^2}$

$S: (x,y) \mapsto (x,y,\sqrt{1-y^2})$

$\vec{r}_x = (1,0,0)$

$D=[0,1]\times[-1,1]$

$\vec{r}_y = \left(0,1,\frac{-y}{\sqrt{1-y^2}}\right)$

At $(0,0,0)$, $\vec{r}_x\times\vec{r}_y = \left(0,\frac{y}{\sqrt{1-y^2}},1\right)$

At $(0,0,0)$, $\vec{r}_x\times\vec{r}_y = (0,0,1)$ pointing outward

$\therefore \hat{n} = \dfrac{(0, y/\sqrt{1-y^2}, 1)}{|\vec{r}_x\times\vec{r}_y|}$ (no need to calculate it)

$$\text{flux} = \iint_D \left(y\sqrt{1-y^2}\,\hat{j} + (1-y^2)\hat{k}\right)\cdot\left(0\hat{i} + \frac{y}{\sqrt{1-y^2}}\hat{j} + \hat{k}\right) dA(x,y)$$

$$= \iint_D y^2 + (1-y^2)\, dA(x,y)$$

$$= \iint_D dA(x,y)$$

$= 2.$ # (no need to use implicit surface)

(IV) Stokes Theorem.

Given a v.f. $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$, its <u>curl</u> is a vector field given by

$$\nabla \times F,\ \text{curl}\ \vec{F} = (P_y - N_z)\hat{i} + (-P_x + M_z)\hat{j} + (N_x - M_y)\hat{k}$$

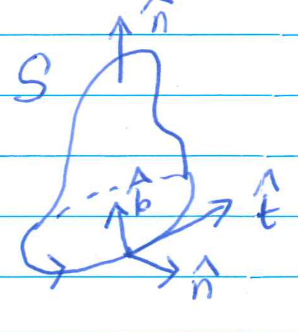
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \quad \text{(symbolically)}$$

<u>Stokes Theorem</u> Let S be an oriented surface in an open region G whose boundary is a closed curve C. Let $\vec{F}$ be a C^2-vector field in G. Then

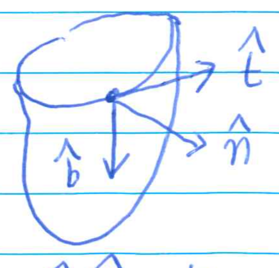
$$\iint_S \nabla \times F \cdot \hat{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}, \quad \text{where}$$

the orientation of C is consistent with the orientation of S.

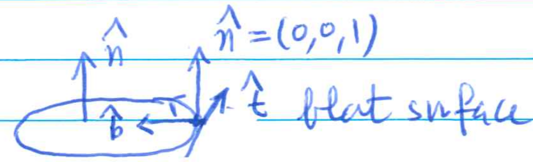
Let $\hat{n}$ be the normal of S at P and $\hat{t}$ be the tangent of C at P. $\hat{n}$ and $\hat{t}$ are consistent if $\hat{n}, \hat{t}, \hat{b}$ form the right hand rule. Here $\hat{b}$ is the "binormal" of S at P, ie, it is perpendicular to $\hat{t}$ and $\hat{n}$, pointing inward S.



$\hat{n}, \hat{t}$ consistent



$\hat{n}, \hat{t}$ not consistent



$\hat{t} = (-1,0,0)$

$\hat{n} = (0,0,1)$

$\hat{b} = (0,-1,0)$

$\hat{n}, \hat{t}$ consistent

eg 6. Calculate the flux of $\vec{F} = y\hat{i} - x\hat{j}$ across the surface $S = x^2+y^2+z^2=9,\ z\geq 0$, with normal pointing upward.

$$\vec{F} = y\hat{i} - x\hat{j}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - 2\hat{k} = -2\hat{k}$$

$$S : \underset{D_3}{(x,y)} \mapsto (x, y, \sqrt{9-x^2-y^2}),$$

$$\vec{r}_x = \left(1, 0, \frac{-x}{\sqrt{\quad}}\right),$$

$$\vec{r}_y = \left(0, 1, \frac{-y}{\sqrt{\quad}}\right)$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -x/\sqrt{\quad} \\ 0 & 1 & -y/\sqrt{\quad} \end{vmatrix} = \left(\frac{-x}{\sqrt{\quad}}, \frac{-y}{\sqrt{\quad}}, 1\right)$$

$\vec{r}_x \times \vec{r}_y$ points upward ($\because$ the last component $= 1 > 0$)

$\therefore$ it is along the $\hat{n}$-direction.

One checks that the anticlockwise direction for

$C : x^2+y^2=9,\ z=0,$

is the tangent direction consistent with S. By Stokes'

$$\text{Flux} : \oint_C \vec{F}\cdot d\vec{r} = \iint_{D_3} \nabla\times\vec{F}\cdot\vec{r}_x\times\vec{r}_y \, dA(x,y)$$

$$= \iint_{D_3} -2\hat{k}\cdot\left(-\frac{x}{\sqrt{9\ldots}}\hat{i} - \frac{y}{\sqrt{\quad}}\hat{j} + \hat{k}\right) dA(x,y)$$

$$= -2\iint_{D_3} dA(x,y)$$

$$= -18\pi \quad \#$$

One may check the result by direct computation.

Let $t \mapsto 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 0\,\hat{k}$, $t \in [0, 2\pi]$

parametrizes C. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (3\sin t\,\hat{i} - 3\cos t\,\hat{j}) \cdot (-3\sin t\,\hat{i} + 3\cos t\,\hat{j})\,dt$$

$$= \int_0^{2\pi} -9\,dt$$

$$= -18\pi \quad \#$$